CROCO – Training Barcelonnette 2023

Advection and diffusion

CROCO
Coastal and Regional Ocean COmmunity model

Reynolds averaged equations

Coastal and Regional Ocean COmmunity model

$$
\begin{aligned}
\frac{D\langle \mathbf{u}_h \rangle}{Dt} + f\mathbf{k}\times\langle \mathbf{u}_h \rangle &= \frac{\boldsymbol{\nabla}_h p}{\rho_0} - \boldsymbol{\nabla}_h \cdot \langle \mathbf{u}'_h \mathbf{u}'_h \rangle - \partial_z \langle w' \mathbf{u}'_h \rangle \\
\partial_z p &= -g\rho' \\
\boldsymbol{\nabla}\cdot\langle \mathbf{u} \rangle &= 0 \\
\frac{D\langle T \rangle}{Dt} &= -\frac{\partial_z Q_s}{\rho_0 C_{p,o}} - \boldsymbol{\nabla}_h \cdot \langle \mathbf{u}'_h T' \rangle - \partial_z \langle w' T' \rangle \\
\frac{D\langle S \rangle}{Dt} &= -\boldsymbol{\nabla}_h \cdot \langle \mathbf{u}'_h S' \rangle - \partial_z \langle w' S' \rangle \\
\rho &= \rho_{\mathrm{eos}}(\langle T \rangle, \langle S \rangle, z)
\end{aligned}
$$

with $\frac{D\langle X \rangle}{Dt} = \partial_t \langle X \rangle + \boldsymbol{\nabla}\cdot\langle X \rangle \langle \mathbf{u} \rangle$

Reynolds averaged equations

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\begin{aligned}
\frac{D\langle \mathbf{u}_h\rangle}{Dt} + f\mathbf{k}\times\langle \mathbf{u}_h\rangle &= \frac{\nabla_h p}{\rho_0} - \nabla_h\cdot\langle \mathbf{u}_h'\mathbf{u}_h'\rangle - \partial_z\langle w'\mathbf{u}_h'\rangle \\
\partial_z p &= -g\rho' \\
\nabla\cdot\langle \mathbf{u}\rangle &= 0 \\
\frac{D\langle T\rangle}{Dt} &= -\frac{\partial_z Q_s}{\rho_0 C_{p,o}} - \nabla_h\cdot\langle \mathbf{u}_h' T'\rangle - \partial_z\langle w'T'\rangle \\
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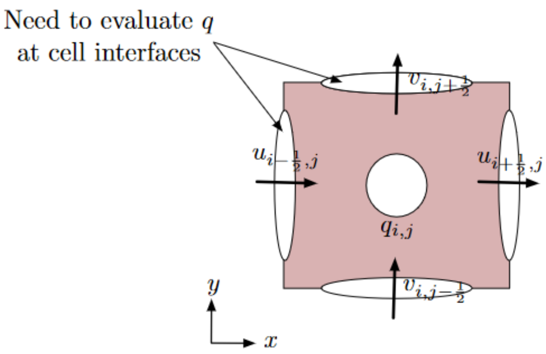
\rho &= \rho_{\text{eos}}(\langle T\rangle, \langle S\rangle, z)
\end{aligned}
$$

with $$\frac{D\langle X\rangle}{Dt} = \partial_t\langle X\rangle + \nabla\cdot\langle X\rangle\langle \mathbf{u}\rangle$$

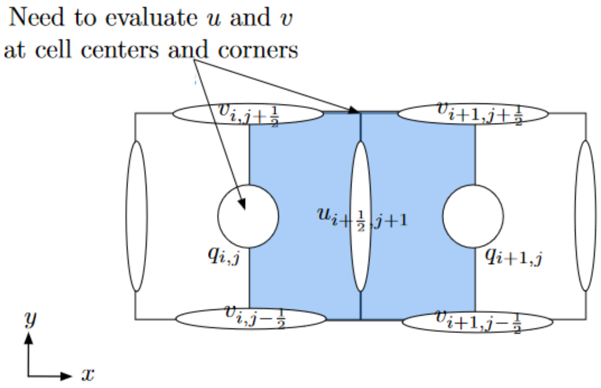
computed with advection schemes

Advection discretisation = interpolation problem

Tracers

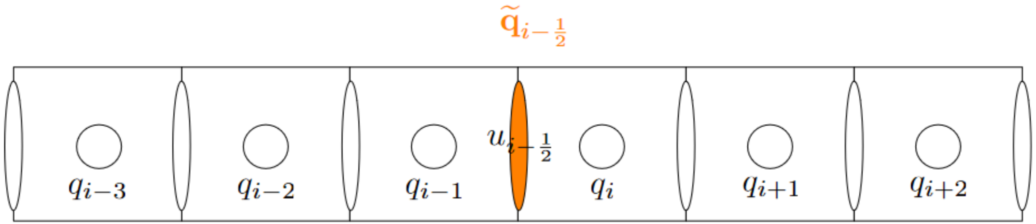


Momentum



Horizontal advection : linear schemes

Horizontal advection [HADV_C2, HADV_UP3, HADV_C4, HADV_UP5, HADV_C6]



$$\partial_x(uq)|_{x=x_i} = \frac{1}{\Delta x_i}\left\{u_{i+1/2}\widetilde{q}_{i+1/2} - u_{i-1/2}\widetilde{q}_{i-1/2}\right\}$$

$$\widetilde{q}^{\,C2}_{i-1/2} = \frac{q_i + q_{i-1}}{2}$$

$$\widetilde{q}^{\,C4}_{i-1/2} = (7/6)\widetilde{q}^{\,C2}_{i-1/2} - (1/12)(q_{i+1} + q_{i-2})$$

$$\widetilde{q}^{\,UP3}_{i-1/2} = \widetilde{q}^{\,C4}_{i-1/2} + \text{sign}(1/12, u_{i-1/2})(q_{i+1} - 3q_i + 3q_{i-1} - q_{i-2})$$

$$\widetilde{q}^{\,C6}_{i-1/2} = (8/5)\widetilde{q}^{\,C4}_{i-1/2} - (19/60)\widetilde{q}^{\,C2}_{i-1/2} + (1/60)(q_{i+2} + q_{i-3})$$

$$\widetilde{q}^{\,UP5}_{i-1/2} = \widetilde{q}^{\,C6}_{i-1/2} - \text{sign}(1/60, u_{i-1/2})(q_{i+2} - 5q_{i+1} + 10q_i - 10q_{i-1} + 5q_{i-2} - q_{i-3})$$

Horizontal advection : linear schemes

Horizontal advection : properties of linear schemes

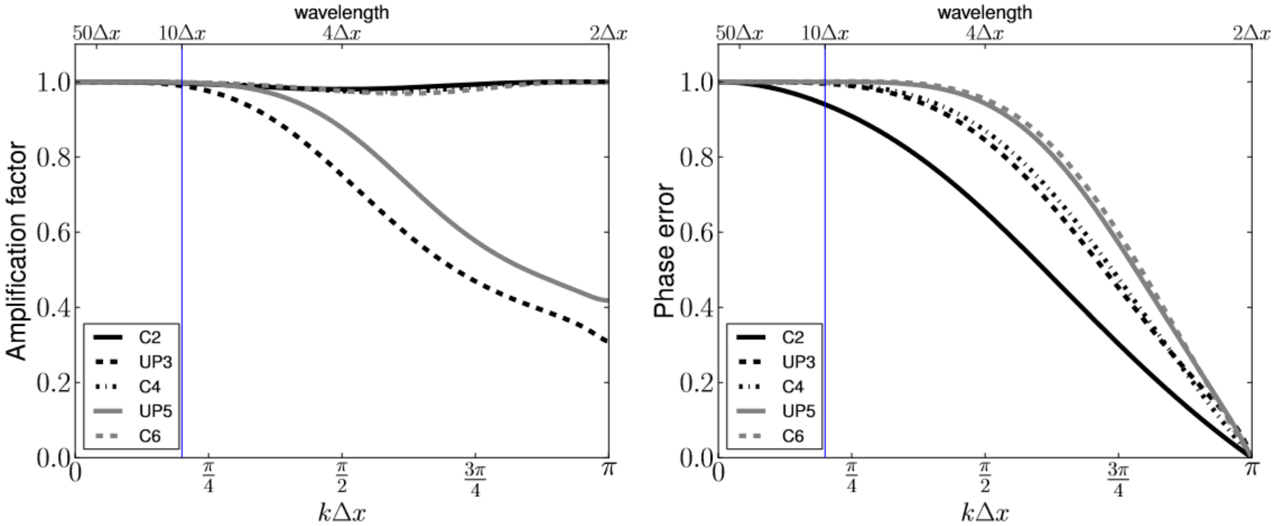
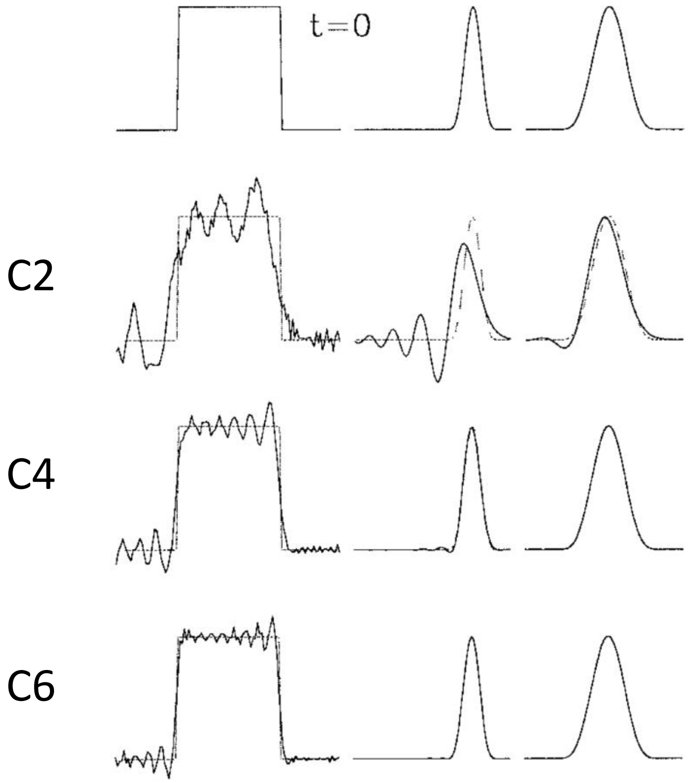


Figure: Amplification error (left) and phase error (right) for linear advection schemes of order 2 to 6.

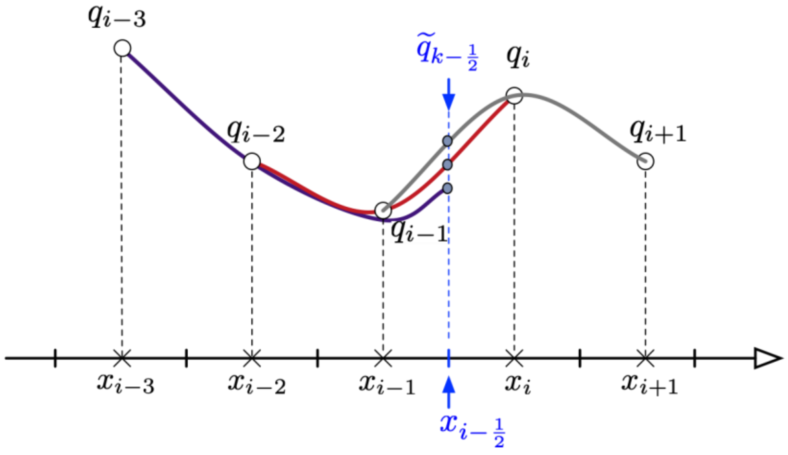
Horizontal advection : linear schemes



- spurious over/under shoots appear where the advected signal varies sharply
- explicit diffusion is needed with even-order schemes
- tend to promote odd-order schemes UP3 and UP5

Horizontal advection : WENO schemes

[HADV_WENO5]



Non-linear weighted average of 3 estimate of the value at interface

$$\widetilde{q}_{k-\frac{1}{2}} = w_0 \widetilde{q}^{(0)}_{k-\frac{1}{2}} + w_1 \widetilde{q}^{(1)}_{k-\frac{1}{2}} + w_2 \widetilde{q}^{(2)}_{k-\frac{1}{2}}$$

1. Convexity
2. ENO (essentially non-oscillatory) property
3. 5th order when q(x) smooth

Do not strictly preserve monotonicity!

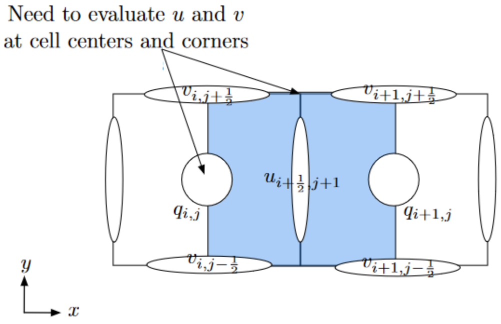
But favored choice with biogeochemistry…

BIO_HADV_WENO5

Horizontal advection

Advection of momentum

$$\partial_t(\mathrm{Hz}u) + \partial_x\left((\mathrm{Hz}\ u)u\right) + \partial_y\left((\mathrm{Hz}\ v)\ u\right) + \dots$$
$$\partial_t(\mathrm{Hz}v) + \partial_x\left((\mathrm{Hz}\ u)v\right) + \partial_y\left((\mathrm{Hz}\ v)\ v\right) + \dots$$



Because of the variable staggering, we need to interpolate both the fluxes and the advected quantity :

$$\left((\widetilde{\mathrm{Hz}\ u})u\right)_{i,j} = (\widetilde{\mathrm{Hz}\ u})^{C4}_{i,j}\ \tilde{u}^{UP3}_{i,j}$$

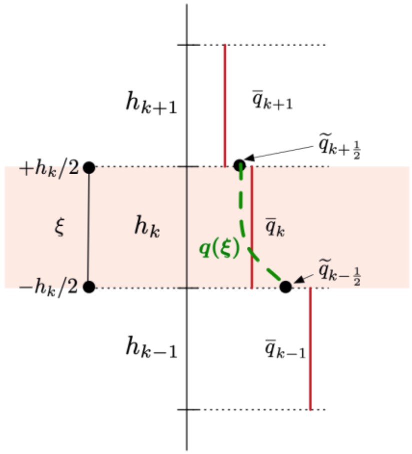
$$\left((\widetilde{\mathrm{Hz}\ v})u\right)_{i+\frac{1}{2},j+\frac{1}{2}} = (\widetilde{\mathrm{Hz}\ v})^{C4}_{i+\frac{1}{2},j+\frac{1}{2}}\ \tilde{u}^{UP3}_{i+\frac{1}{2},j+\frac{1}{2}}$$

Vertical advection : SPLINE reconstruction

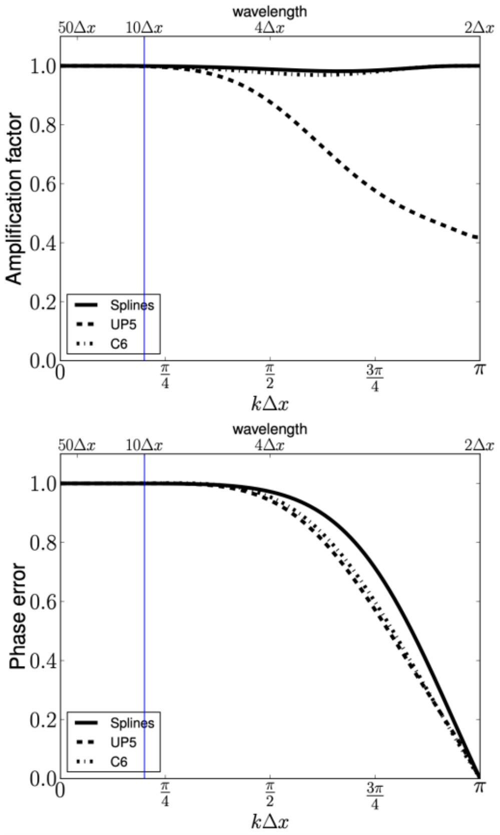
[TS_VADV_SPLINES, UV_VADV_SPLINES]

Fluxes are obtained by solving the tridiagonal problem

$$\mathrm{Hz}_{k+1}\widetilde{q}_{k-\frac{1}{2}} + 2(\mathrm{Hz}_k + \mathrm{Hz}_{k+1})\widetilde{q}_{k+\frac{1}{2}} + \mathrm{Hz}_k\widetilde{q}_{k+\frac{3}{2}} = 3(\mathrm{Hz}_k\overline{q}_{k+1} + \mathrm{Hz}_{k+1}\overline{q}_k)$$



$$q(\xi) = \overline{q}_k + \frac{\widetilde{q}_{k+\frac{1}{2}} - \widetilde{q}_{k-\frac{1}{2}}}{h_k}\xi + 6\left(\frac{\widetilde{q}_{k+\frac{1}{2}} + \widetilde{q}_{k-\frac{1}{2}}}{2} - \overline{q}_k\right)\left(\frac{\xi^2}{h_k^2} - \frac{1}{12}\right)$$



Vertical advection : Adaptive implicit SPLINE reconstruction

Coastal and Regional Ocean COmmunity model

[VADV_ADAPT_IMP] [Shchepetkin, 2015]

$$\Omega = \mathbf{\Omega}^{(e)} + \mathbf{\Omega}^{(i)}, \qquad \mathbf{\Omega}^{(e)} = \frac{\Omega}{f(\alpha^z_{adv}, \alpha_{max})}, \qquad f(\alpha^z_{adv}, \alpha_{max}) = \begin{cases} 1, & \alpha^z_{adv} \leq \alpha_{max} \\ \alpha/\alpha_{max}, & \alpha^z_{adv} > \alpha_{max} \end{cases}$$

$\rightarrow$ $\mathbf{\Omega}^{(e)}$ integrated with an explicit scheme (whose CFL is α_{max})

$\rightarrow$ $\mathbf{\Omega}^{(i)}$ integrated with an upwind Euler implicit scheme (unconditionnaly stable)

configuration	resolution	old dt	new dt
BENGUELA (Penven et al)	25 km	6300s	7140s
OMAN (Vic et al)	2 km	160s	470s

- No control on errors
- Need a linear scheme for the explicit part

Vertical advection : AKIMA

[TS_VADV_AKIMA]

$$\widetilde{q}^{C4}_{k-\frac{1}{2}} = (\frac{7}{6})\widetilde{q}^{C2}_{k-\frac{1}{2}} - (\frac{1}{12})(q_{k+1} + q_{k-2})$$

$$= \widetilde{q}^{C2}_{k-\frac{1}{2}} - \frac{1}{6}\left(d_k - d_{k-1}\right), \qquad d_k = \frac{\Delta q_{k+\frac{1}{2}} + \Delta q_{k-\frac{1}{2}}}{2}$$

With AKIMA scheme , algebraic average of elementary differences is replaced by harmonic average

$$d_k = \begin{cases} \dfrac{2}{\dfrac{1}{\Delta q_{k+\frac{1}{2}}} + \dfrac{1}{\Delta q_{k-\frac{1}{2}}}} & \text{si } \Delta q_{k+\frac{1}{2}} \Delta q_{k-\frac{1}{2}} > 0 \\ 0 & \text{sinon} \end{cases}$$

=> better control of over/under- shoots

Numerical schemes: Advection

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- Available schemes in CROCO

Equation	horizontal	vertical
Moment 3D (UV_ADV) (+ eq W_ en NBQ)	UV_HADV_TVD UV_HADV_C2 **UV_HADV_UP3** UV_HADV_C4 UV_HADV_UP5 UV_HADV_C6 UV_HADV_WENO5	UV_VADV_TVD UV_VADV_C2 **UV_VADV_SPLINES** UV_VADV_WENO5
Traceurs	**TS_HADV_UP3** TS_HADV_RSUP3 TS_HADV_C4 TS_HADV_WENO5 TS_HADV_UP5 TS_HADV_C6 TS_HADV_RSUP5	TS_VADV_TVD TS_VADV_C2 TS_VADV_SPLINES **TS_VADV_AKIMA** TS_VADV_WENO5
Moment 2D	**M2_HADV_UP3** M2_HADV_C2	

Advices for choosing:

- Not choosing => take default
- Look at sources of errors
- Consider the physics (scales, need of positivity…)
- Consider computation time

Reynolds averaged equations

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\rho &= \rho_{\text{eos}}(\langle T\rangle, \langle S\rangle, z)
\end{aligned}
$$

represented by (hyper-)diffusion operators

with $\frac{D\langle X\rangle}{Dt} = \partial_t \langle X\rangle + \boldsymbol{\nabla}\cdot\langle X\rangle\langle \mathbf{u}\rangle$

Viscous operators

[UV_VIS2]

$$-\nabla_h \cdot \langle \mathbf{u}'_h \mathbf{u}'_h \rangle = \frac{1}{\mathrm{Hz}} \nabla_h \cdot (A_M \, \mathrm{Hz} \, \boldsymbol{\sigma}) \,, \qquad A_M \leftrightarrow \mathrm{visc2}$$

$$\boldsymbol{\sigma}(\mathbf{u}_h) = \begin{pmatrix} \partial_x u - \partial_y v & \partial_y u + \partial_x v \\ \partial_x v + \partial_y u & -(\partial_x u - \partial_y v) \end{pmatrix} \qquad \mathbf{u}_h = [u, v]$$

satisfy : - momentum conservation
- angular momentum conservation
- strictly dissipative term

[UV_VIS4] same logics applied twice

$$-\nabla_h \cdot \langle \mathbf{u}'_h \mathbf{u}'_h \rangle = -\frac{1}{\mathrm{Hz}} \nabla_h \cdot (B_M \, \mathrm{Hz} \, \boldsymbol{\sigma}') \,, \quad (B_M \leftrightarrow \mathrm{visc4})$$

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma}(\nabla_h \cdot \boldsymbol{\sigma}(\mathbf{u}_h))$$

Viscous/Diffusion operators : Smagorinsky closures

[UV_VIS_SMAGO, UV_VIS2]

$$A_M = C_M \, (\Delta x \Delta y) \sqrt{(\partial_x u)^2 + (\partial_y v)^2 + 2(\partial_y u + \partial_x v)^2} \qquad C_M = \frac{1}{10}$$

[TS_DIF_SMAGO, UV_VIS2]

$$A_S = C_S \, (\Delta x \Delta y) \sqrt{(\partial_x u)^2 + (\partial_y v)^2 + 2(\partial_y u + \partial_x v)^2} \qquad C_S = \frac{1}{12}$$

Rotated (hyper-)diffusion

Coastal and Regional Ocean COmmunity model

[TS_MIX_ISO, TS_MIX_GEO]

Under small slope approximation (i.e. $\frac{\|\nabla_h \rho\|}{\partial_z \rho} \ll 1$)

TS_DIF2

$$-\nabla_h \cdot \langle \mathbf{u}'_h X' \rangle = \nabla \cdot (\mathbf{R} \nabla X), \qquad \mathbf{R} = \begin{pmatrix} A_x & 0 & A_x \alpha_x \\ 0 & A_y & A_y \alpha_y \\ A_x \alpha_x & A_y \alpha_y & A_x \alpha_x^2 + A_y \alpha_y^2 \end{pmatrix}$$

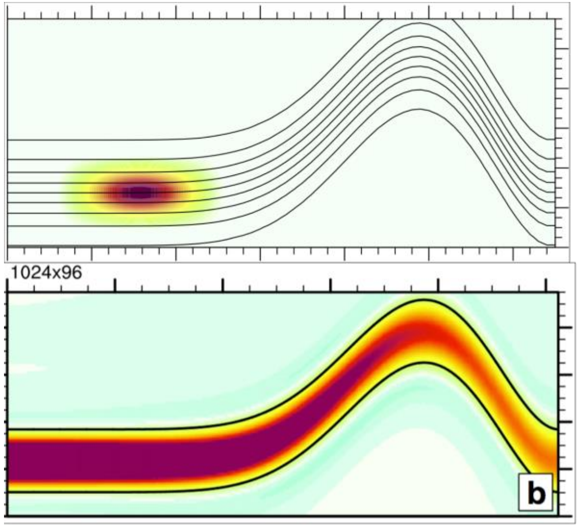
$$\alpha_m = -\left(\frac{\partial_m \rho}{\partial_z \rho}\right), \; A_x \leftrightarrow \text{diff3u}, \; A_y \leftrightarrow \text{diff3v}$$

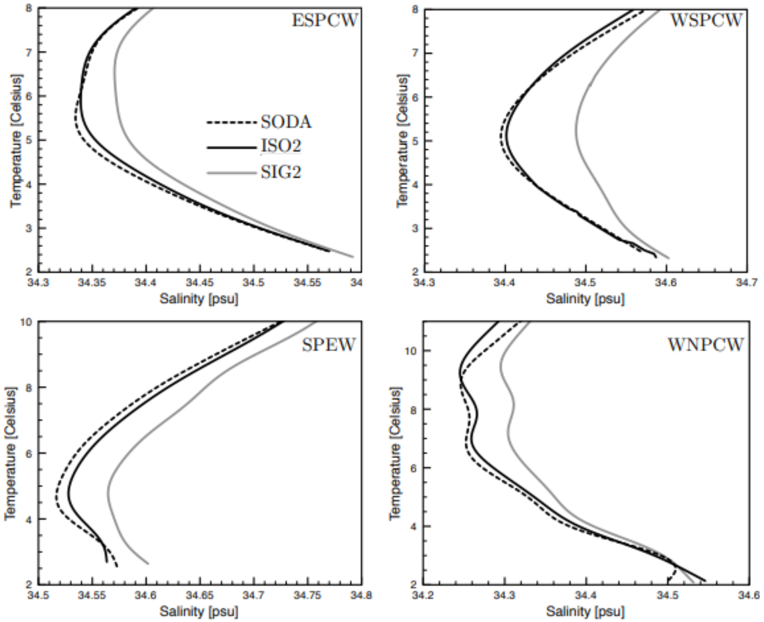
TS_DIF4

$$-\nabla_h \cdot \langle \mathbf{u}'_h X' \rangle = -\nabla \cdot (\mathbf{R}' \nabla (\nabla \cdot (\mathbf{R}' \nabla X))), \quad \mathbf{R}' = \begin{pmatrix} \sqrt{B_x} & 0 & \sqrt{B_x} \alpha_x \\ 0 & \sqrt{B_y} & \sqrt{B_y} \alpha_y \\ \sqrt{B_x} \alpha_x & \sqrt{B_y} \alpha_y & \sqrt{B_x} \alpha_x^2 + \sqrt{B_y} \alpha_y^2 \end{pmatrix}$$

Rotated (hyper-)diffusion

[TS_MIX_ISO, TS_MIX_GEO]





For more details

https://croco-ocean.gitlabpages.inria.fr/croco_doc/index.html