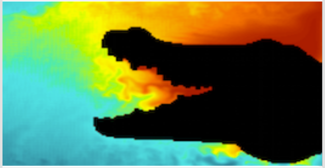
CROCO

Coastal and Regional Ocean
COmmunity model



Time-stepping of the CROCO hydrostatic code
and its impact on code structure (and on advection schemes)

Nicolas Ducousso – Shom, Brest, France

Florian Lemarié – Inria (EPI AIRSEA), Laboratoire Jean Kuntzmann, Grenoble, France

General context

Reynolds averaged primitive equations ($X = X' + \langle X \rangle$)

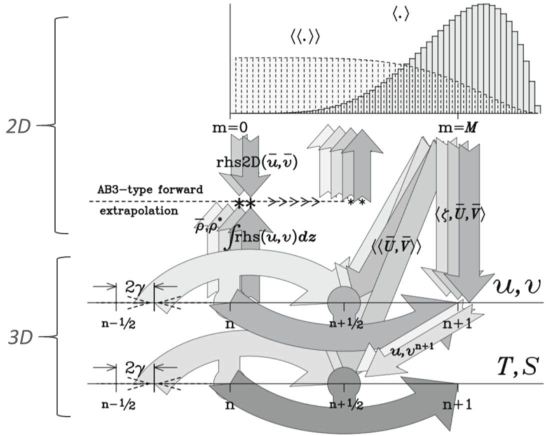
$$\begin{aligned}
\frac{D\left\langle \mathbf{u}_h \right\rangle}{Dt} + f\mathbf{k} \times \left\langle \mathbf{u}_h \right\rangle &= \frac{\boldsymbol{\nabla}_h p}{\rho_0} - \boldsymbol{\nabla}_h \cdot \left\langle \mathbf{u}_h' \mathbf{u}_h' \right\rangle - \partial_z \left\langle w' \mathbf{u}_h' \right\rangle \\
\partial_z p &= -g\rho' \\
\boldsymbol{\nabla} \cdot \left\langle \mathbf{u} \right\rangle &= 0 \\
\frac{D\left\langle T \right\rangle}{Dt} &= -\frac{\partial_z Q_s}{\rho_0 C_{p,o}} - \boldsymbol{\nabla}_h \cdot \left\langle \mathbf{u}_h' T' \right\rangle - \partial_z \left\langle w' T' \right\rangle \\
\frac{D\left\langle S \right\rangle}{Dt} &= -\boldsymbol{\nabla}_h \cdot \left\langle \mathbf{u}_h' S' \right\rangle - \partial_z \left\langle w' S' \right\rangle \\
\rho &= \rho_{\text{eos}}(\left\langle T \right\rangle, \left\langle S \right\rangle, |z|)
\end{aligned}$$

The objective : effectively address the scale separation problem between

- the fast evolving dynamics related to the ocean/atmosphere difference of density and
- the slow evolving dynamics related to the internal ocean stratification.

”Schematic view” of the time-stepping

Shchepetkin A. and J. McWilliams : ***The Regional Oceanic Modeling System: A split-explicit, free-surface, topography-following-coordinate ocean model*****, Ocean Model., 2005**



- mode-splitting technique
- split-explicit approach
- forward-backward feedbacks
- high order numerics

Content

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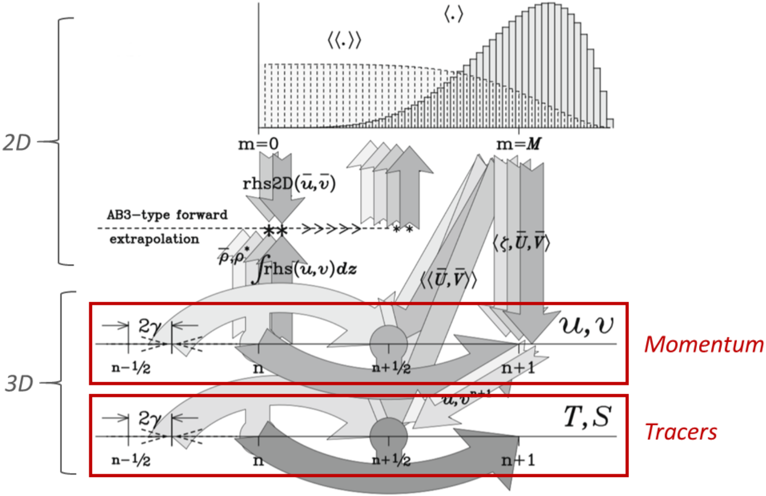
3D momentum and tracers

3D momentum and tracers



3D momentum and tracers : time stepping

Predictor-corrector : Leapfrog (LF) with 3rd-order Adams-Moulton (AM) interpolation ⇒ LF-AM3

For a given quantity q

$$\begin{cases} q^{n+1,\star} = q^{n-1} + 2\Delta t \,\mathrm{RHS}\,\{q^n\} & \text{(LF)} \\ q^{n+\frac{1}{2}} = ({}^5\!/_{12})\, q^{n+1,\star} + ({}^2\!/_3)\, q^n - ({}^1\!/_{12})\, q^{n-1} & \text{(AM3)} \\ q^{n+1} = q^n + \Delta t\,\mathrm{RHS}\left\{q^{n+\frac{1}{2}}\right\} & \text{(corrector)} \end{cases}$$

Compact version used in the code :

$$\begin{aligned} q^{n+\frac{1}{2}} &= ({}^1\!/_2 - \gamma)\, q^{n-1} + ({}^1\!/_2 + \gamma)\, q^n + (1-\gamma)\Delta t\,\mathrm{RHS}\,\{q^n\} \\ q^{n+1} &= q^n + \Delta t\,\mathrm{RHS}\left\{q^{n+\frac{1}{2}}\right\} \end{aligned}$$

with $\gamma = {}^1\!/_6$ (cf pre_step3d.F).

Impact on the structure of the code

→ LFAM3 is conditionally stable for oscillation-like and friction-like processes

- Depending on processes, use LFAM3 or an Euler step (forward or backward)

→ Some terms are computed twice per time-steps

- 3D Advection
- Equation of state
- Pressure gradient
- Continuity equation
- Coriolis term

→ Some terms are computed once using an Euler step

- Physical parameterization of vertical mixing
- Rotated diffusion
- Viscous terms and diffusion in general

Structure of the arrays for 3D variables

→ Need to store 3 time indices

```
real u(GLOBAL_2D_ARRAY,N,3)
real v(GLOBAL_2D_ARRAY,N,3)
real t(GLOBAL_2D_ARRAY,N,3,NT)
```

→ time indices

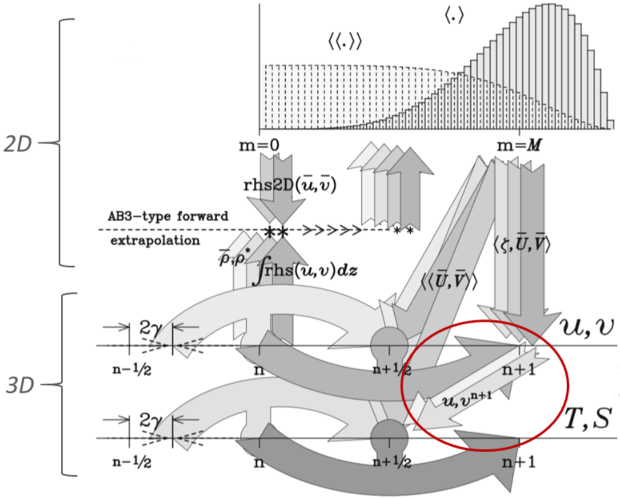
Predictor

```
nstp=1+mod(iic-ntstart,2) <-- nstp = 1 or 2
nrhs=nstp       <-- n
nnew=3          <-- n+1/2
indx=3-nstp     <-- n-1
```

Corrector

```
nrhs=3          <-- n+1/2
nnew=3-nstp     <-- n+1
```

3D momentum and tracers : momentum-tracers coupling



Momentum-tracer coupling

3D momentum and tracers : momentum-tracers coupling

$$\begin{cases} \partial_z w + \partial_x u & = & 0 \\ \partial_z p + \rho g & = & 0 \\ \partial_t u + ({}^1/_{\rho_0})\partial_x p & = & 0 \\ \partial_t \rho + \partial_z (w\rho) & = & 0 \end{cases}$$

3D momentum and tracers : momentum-tracers coupling

Predictor

$$\partial_x p^n = g\,\partial_x\left(\int_z^0 \rho^n\,dz\right) \quad\rightarrow\quad u^{n+1/2} = (1/2-\gamma)\,u^{n-1} + (1/2+\gamma)\,u^n + (1-\gamma)\,\frac{\Delta t}{\rho_0}\,(\partial_x p^n)$$

$$w^n = -\int_{-H}^{z} \partial_x u^n\,dz' \quad\rightarrow\quad \rho^{n+1/2} = (1/2-\gamma)\,\rho^{n-1} + (1/2+\gamma)\,\rho^n + (1-\gamma)\Delta t\;\partial_z(w^n\rho^n)$$

Corrector

$$\partial_x p^{n+1/2} = g\,\partial_x\left(\int_z^0 \rho^{n+1/2}\,dz\right) \quad\rightarrow\quad u^{n+1} = u^n + (\Delta t/\rho_0)\,(\partial_x p^{n+1/2})$$

$$w^{n+1/2} = -\int_{-H}^{z} \partial_x\left\{\frac{3u^{n+1/2}}{4} + \frac{u^n + u^{n+1}}{8}\right\}dz' \quad\rightarrow\quad \rho^{n+1} = \rho^n + \Delta t\;\partial_z(w^{n+1/2}\rho^{n+1/2})$$

Consequences :

- 3D-momentum integrated before the tracers in the corrector
- 2 evaluations of the pressure gradient per time-step
- 3 evaluations of the continuity equation per time-step

3D momentum and tracers : momentum-tracers coupling

Predictor

$$\partial_x p^n = g\,\partial_x \left(\int_z^0 \rho^n\, dz\right) \quad \rightarrow \quad u^{n+1/2} = (1/2-\gamma)\,u^{n-1} + (1/2+\gamma)\,u^n + (1-\gamma)\,\frac{\Delta t}{\rho_0}\,(\partial_x p^n)$$

$$w^n = -\int_{-H}^{z} \partial_x u^n\, dz' \quad \rightarrow \quad \rho^{n+1/2} = (1/2-\gamma)\,\rho^{n-1} + (1/2+\gamma)\,\rho^n + (1-\gamma)\Delta t\;\partial_z(w^n \rho^n)$$

Corrector

$$\partial_x p^{n+1/2} = g\,\partial_x \left(\int_z^0 \rho^{n+1/2} dz\right) \quad \rightarrow \quad u^{n+1} = u^n + (\Delta t/\rho_0)\,(\partial_x p^{n+1/2})$$

$$w^{n+1/2} = -\int_{-H}^{z} \partial_x \left\{\frac{3u^{n+1/2}}{4} + \frac{u^n + u^{n+1}}{8}\right\} dz' \quad \rightarrow \quad \rho^{n+1} = \rho^n + \Delta t\;\partial_z(w^{n+1/2}\rho^{n+1/2})$$

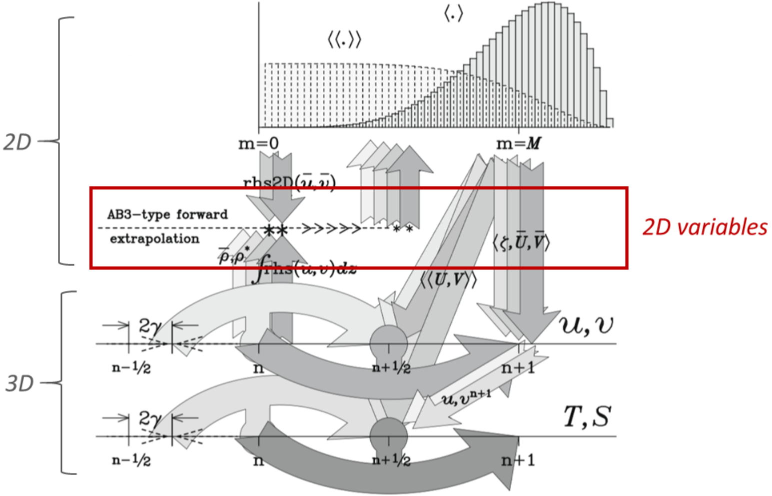
Benefits :

- increase the stability range for the integration of internal waves

2

2D ”barotropic” variables

2D barotropic variables



2D barotropic variables : time-stepping

3rd-order Adams-Bashforth (AB) with forward-backward feedback ⇒ Generalized Forward-Backward

- AB3-type extrapolation

$$\begin{aligned} D^{\mathbf{m+1/2}} &= H + (3/2+\beta)\,\zeta^{\mathbf{m}} - (1/2+2\beta)\,\zeta^{\mathbf{m\text{-}1}} + \beta\zeta^{\mathbf{m\text{-}2}} \\ \overline{u}^{\mathbf{m+1/2}} &= (3/2+\beta)\,\overline{u}^{\mathbf{m}} - (1/2+2\beta)\,\overline{u}^{\mathbf{m\text{-}1}} + \beta\overline{u}^{\mathbf{m\text{-}2}} \end{aligned}$$

- Integration of ζ

$$\zeta^{\mathbf{m+1}} = \zeta^{\mathbf{m}} - \Delta\tau\,\partial_x(D^{\mathbf{m+1/2}}\overline{u}^{\mathbf{m+1/2}})$$

- AM4 interpolation

$$\zeta^{\star} = (1/2+\gamma+2\varepsilon)\zeta^{\mathbf{m+1}} + (1/2-2\gamma-3\varepsilon)\zeta^{\mathbf{m}} + \gamma\zeta^{\mathbf{m\text{-}1}} + \varepsilon\zeta^{\mathbf{m\text{-}2}}$$

- Integration of $\bar{u}$

$$\overline{u}^{\mathbf{m+1}} = \frac{1}{D^{\mathbf{m+1}}}\left[D^{\mathbf{m}}\overline{u}^{\mathbf{m}} + \Delta\tau\ \mathrm{RHS2D}(D^{\mathbf{m+1/2}}, \overline{u}^{\mathbf{m+1/2}}, \zeta^{\star})\right]$$

Structure of the arrays for barotropic variables

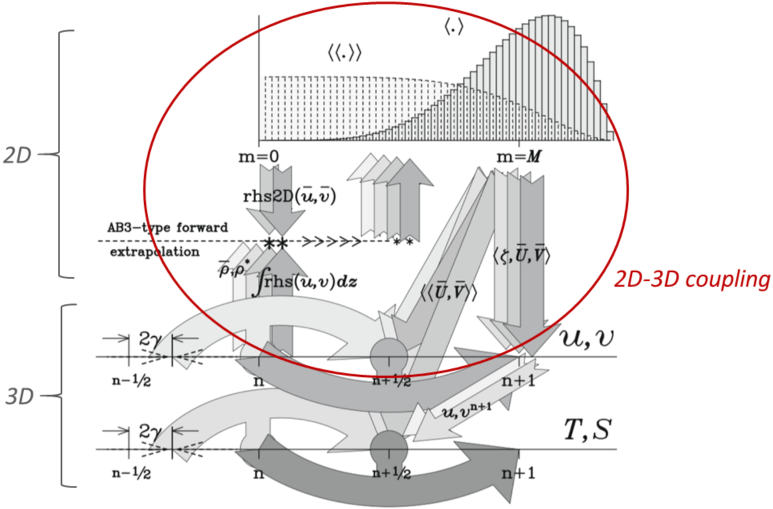
→ Need to store 4 time indices

```
real zeta(GLOBAL_2D_ARRAY,4)
real ubar(GLOBAL_2D_ARRAY,4)
real vbar(GLOBAL_2D_ARRAY,4)
```

→ Time indices

```
kold    <-- m-2
kbak    <-- m-1
kstp    <-- m
knew    <-- m+1
```

2D-3D coupling

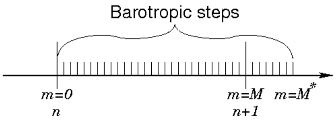


2D-3D coupling [M2_FILTER_POWER]

- Slow forcing term of the 2D by the 3D is extrapolated

$$\mathcal{F}_{3D}^{n+1/2} = \left\{ \int \mathrm{rhs}(u,v)dz - \mathrm{rhs2D}(\bar{u},\bar{v}) \right\}^{n+1/2} = \mathrm{Extrap}(\mathcal{F}_{3D}^{n}, \mathcal{F}_{3D}^{n-1}, \mathcal{F}_{3D}^{n-2})$$

- 2D integration from n to $n + M^* \Delta\tau$ $(M^* \leq 1.5M)$



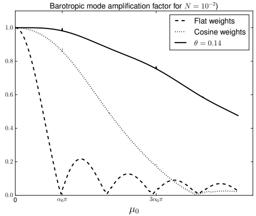
- Low-pass filtered 2D variables are needed to set and correct 3D variables
  - require a dual set of averaging filters $\langle . \rangle^{n+1}$ and $\langle\langle . \rangle\rangle^{n+1/2}$
  - $\langle \zeta \rangle^{n+1} \rightarrow$ update of the vertical grid
  - $\langle U \rangle^{n+1} \rightarrow$ correction of 3D velocities at time $n+1$
  - $\langle\langle U \rangle\rangle^{n+1/2} \rightarrow$ correction of 3D velocities at time $n + 1/2$

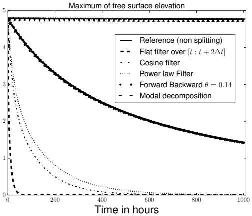
2D-3D coupling : the alternative [M2_FILTER_NONE]

Demange J., L. Debreu, P. Marchesiello, F. Lemarié, E. Blayo (2019) : *Stability analysis of split-explicit free surface ocean models: implication of the depth-independent barotropic mode approximation.*, JCP, 398, 108875.

Motivation: Averaging filters → excessive dissipation in the 2D

Objective: put the minimum amount of dissipation to stabilize the splitting





⇒ diffusion is introduced within the barotropic time-stepping rather than averaging filters

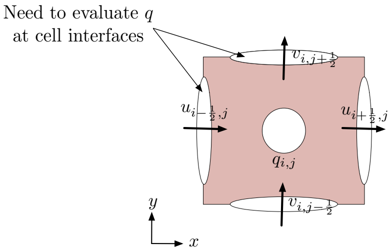
- May require to increase $\mathrm{NDTFAST} = \Delta t_{\mathrm{3D}}/\Delta t_{\mathrm{2D}}$
- Systematically more efficient than averaging filters

3

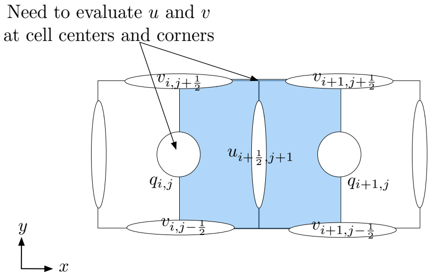
Elements on advection schemes

Advection discretisation = interpolation problem

Tracers



Momentum



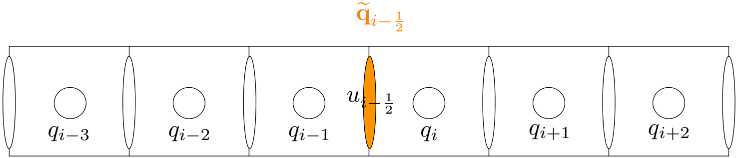
Available advection schemes

Equation	horizontal	vertical
3D momentum	UV_HADV_WENO5	UV_VADV_WENO5
	UV_HADV_C6	UV_VADV_C6
	UV_HADV_UP5	UV_VADV_UP5
	UV_HADV_C4	UV_VADV_C4
	UV_HADV_UP3	UV_VADV_SPLINES
	UV_HADV_C2	UV_VADV_C2
2D momentum	M2_HADV_UP3	-
	M2_HADV_C2	-

Available advection schemes

Equation	horizontal	vertical
Tracers	TS_HADV_WENO5	TS_VADV_WENO5
	TS_HADV_C6	TS_VADV_AKIMA
	TS_HADV_UP5	TS_VADV_SPLINES
	TS_HADV_C4	TS_VADV_C4
	TS_HADV_UP3	TS_VADV_C2
	TS_HADV_RSUP5	TS_VADV_FCT
	TS_HADV_RSUP3	

Horizontal advection [HADV_C2, HADV_UP3, HADV_C4, HADV_UP5, HADV_C6]



$$\partial_x (uq)\big|_{x=x_i} = \frac{1}{\Delta x_i} \left\{ u_{i+1/2} \widetilde{q}_{i+1/2} - u_{i-1/2} \widetilde{q}_{i-1/2} \right\}$$

$$
\begin{aligned}
\widetilde{q}^{\,\mathrm{C2}}_{i-1/2} &= \frac{q_i + q_{i-1}}{2} \\
\widetilde{q}^{\,\mathrm{C4}}_{i-1/2} &= ({}^{7}/{}_{6})\widetilde{q}^{\,\mathrm{C2}}_{i-1/2} - ({}^{1}/{}_{12})(q_{i+1} + q_{i-2}) \\
\widetilde{q}^{\,\mathrm{UP3}}_{i-1/2} &= \widetilde{q}^{\,\mathrm{C4}}_{i-1/2} + \mathrm{sign}({}^{1}/{}_{12}, u_{i-1/2})(q_{i+1} - 3q_i + 3q_{i-1} - q_{i-2}) \\
\widetilde{q}^{\,\mathrm{C6}}_{i-1/2} &= ({}^{8}/{}_{5})\widetilde{q}^{\,\mathrm{C4}}_{i-1/2} - ({}^{19}/{}_{60})\widetilde{q}^{\,\mathrm{C2}}_{i-1/2} + ({}^{1}/{}_{60})(q_{i+2} + q_{i-3}) \\
\widetilde{q}^{\,\mathrm{UP5}}_{i-1/2} &= \widetilde{q}^{\,\mathrm{C6}}_{i-1/2} - \mathrm{sign}({}^{1}/{}_{60}, u_{i-1/2})(q_{i+2} - 5q_{i+1} + 10q_i - 10q_{i-1} + 5q_{i-2} - q_{i-3})
\end{aligned}
$$

Horizontal advection : properties of linear schemes

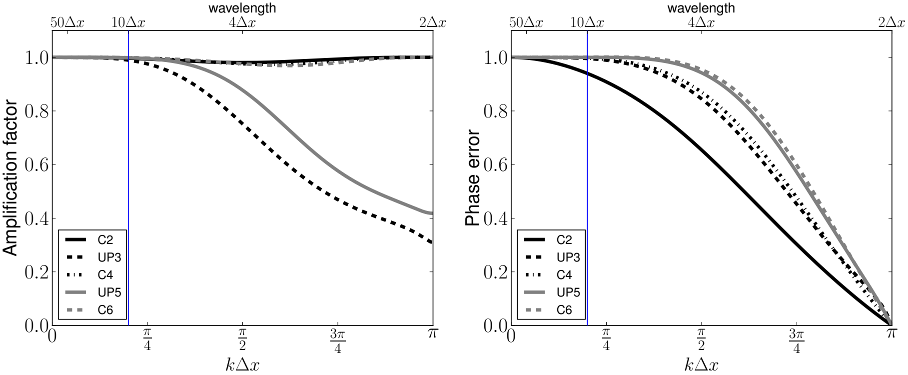


Figure: Amplification error (left) and phase error (right) for linear advection schemes of order 2 to 6.

⇒ explicit diffusion/viscosity is needed with even-ordered schemes

Rotated Split UP3 [TS_HADV_RSUP3]

$$\widetilde{q}^{\,\mathrm{UP3}}_{i-1/2} \quad = \quad \widetilde{q}^{\,\mathrm{C4}}_{i-1/2} + \mathbf{sign}(^1/_{12}, u_{i-1/2})(q_{i+1} - 3q_i + 3q_{i-1} - q_{i-2})$$

Split-UP3

→ $\widetilde{q}^{\,\mathrm{C4}}_{i-1/2}$ treated in the LF-AM3

→ The red term is treated as an explicit diffusion (i.e. with Euler forward)

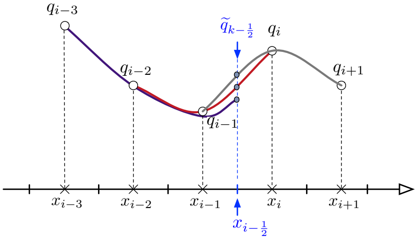
Rotated Split-UP3

→ The diffusive term is rotated in the isoneutral direction

- small slope assumption
- non-monotonic diffusion (additional dispersive cross-terms)
- assumes low frequency evolution of isopycnals
- a rotation along geopotentials could be a good alternative for high-resolution applications

WENO5 Scheme (e.g. Acker et al., 2016) [TS_HADV_WENO5]

Nonlinear weighting between 3 evaluations of interfacial values



$$\widetilde{q}_{k-1/2} = w_0 \widetilde{q}^{(0)}_{k-1/2} + w_1 \widetilde{q}^{(1)}_{k-1/2} + w_2 \widetilde{q}^{(2)}_{k-1/2}$$

1. Convexity ($\sum_{j=0}^{2} w_j = 1$)
2. ENO property (essentially non-scillatory)
3. fifth-order if q(x) is smooth

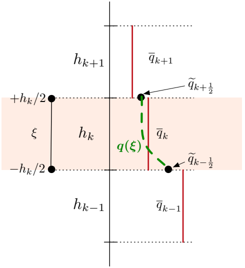
- The scheme satisfies the ENO property (*Total Variation Bounded* property)
- **Warning:** no monotonicity-preserving property !!!

⇒ Best choice for biogeochemical tracers

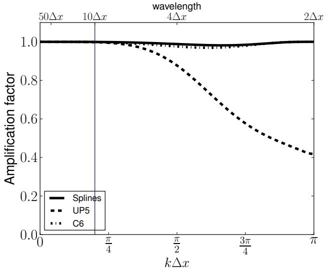
Vertical advection : spline reconstruction [TS_VADV_SPLINES, UV_VADV_SPLINES]

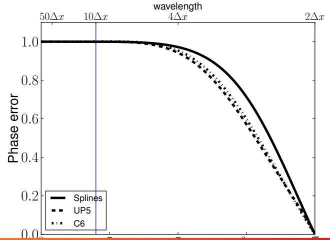
The flux are obtained as a solution of a tridiagonal problem

$$\mathrm{Hz}_{k+1}\widetilde{q}_{k-1/2} + 2(\mathrm{Hz}_k + \mathrm{Hz}_{k+1})\widetilde{q}_{k+1/2} + \mathrm{Hz}_k\widetilde{q}_{k+3/2} = 3(\mathrm{Hz}_k\overline{q}_{k+1} + \mathrm{Hz}_{k+1}\overline{q}_k)$$



$$q(\xi) = \overline{q}_k + \frac{\widetilde{q}_{k+\frac{1}{2}} - \widetilde{q}_{k-\frac{1}{2}}}{h_k}\xi + 6\left(\frac{\widetilde{q}_{k+\frac{1}{2}} + \widetilde{q}_{k-\frac{1}{2}}}{2} - \overline{q}_k\right)\left(\frac{\xi^2}{h_k^2} - \frac{1}{12}\right)$$





Vertical advection : semi-implicit formulation

[VADV_ADAPT_IMP]

Idea : split the vertical velocity Ω in an explicit and implicit contribution [Shchepetkin, 2015]

$$\Omega = \boldsymbol{\Omega}^{(\mathbf{e})} + \boldsymbol{\Omega}^{(\mathbf{i})}, \qquad \boldsymbol{\Omega}^{(\mathbf{e})} = \frac{\Omega}{f(\alpha^z_{\mathrm{adv}}, \alpha_{\max})}, \quad f(\alpha^z_{\mathrm{adv}}, \alpha_{\max}) = \begin{cases} 1, & \alpha^z_{\mathrm{adv}} \leq \alpha_{\max} \\ \alpha/\alpha_{\max}, & \alpha^z_{\mathrm{adv}} > \alpha_{\max} \end{cases}$$

→ $\boldsymbol{\Omega}^{(\mathbf{e})}$ integrated with an explicit scheme (with CFL $\alpha_{\max}$)

→ $\boldsymbol{\Omega}^{(\mathbf{i})}$ integrated with an implicit upwind Euler scheme

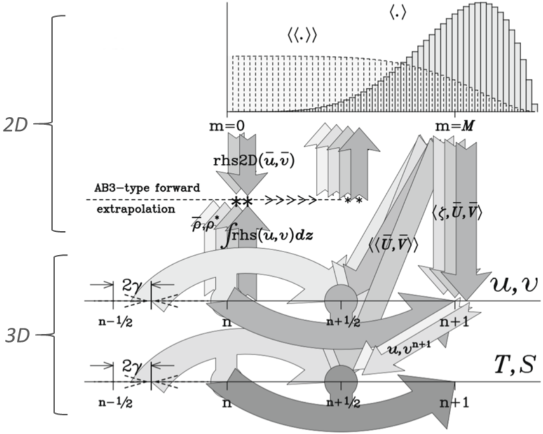
Configuration	Resolution	old Δt	new Δt
BENGUELA [Penven et al.]	25 km	6300 s	7140 s
OMAN [Vic et al.]	2 km	160 s	470 s

▷ Implemented only with Spline reconstruction for the explicit part

4

Flow-chart of the CROCO time-stepping

Structure of the step() subroutine



```
call prestep3D_thread()
call step2d_thread()
call step3D_uv_thread()
call step3D_t_thread()
```

pre_step3D_thread()

- **rho_eos :** computation of the density at time n

$$\rho'^{\,n} = \rho_{\text{EOS}}(T^n, S^n)$$

- **set_HUV :** computation of the volumetric fluxes (i.e. $U^n = \Delta y \Delta z^n u^n$)

$$\mathrm{HUon}_k^n = \mathrm{Hz}_k^n \; \Delta y \; \mathrm{u}_k^n$$

- **omega :** computation of the vertical volumetric fluxes via the continuity equation

$$\mathrm{W}_{k+1/2}^n = -\sum_{k'=1}^{k} (\text{div } \mathrm{HUon})_{k'}^n + \frac{z_{k+1/2}^n - H}{\zeta^n - H} \sum_{k=1}^{N} (\text{div } \mathrm{HUon})_k^n$$

- **prsgrd :** Computation of the pressure gradient and add it to the 3D rhs ru

$$\mathrm{ru}_k = \left.\frac{\partial \mathrm{P}_k^n}{\partial x}\right|_z$$

- **rhs3d :** Computation of the rhs for 3D momentum at time n
  - → Coriolis term

$$\mathrm{ru}^n = \mathrm{ru}^n + \mathrm{Coriolis}$$

  - → Horizontal advection

$$\mathrm{ru}^n = \mathrm{ru}^n + \mathrm{2D\ advection}$$

  - → Vertical advection

$$\mathrm{ru}^n = \mathrm{ru}^n + \mathrm{vertical\ advection}$$

Computation of the forcing term for momentum in the barotropic mode (add the difference between the surface and bottom friction)

$$\mathrm{rufrc}^n = \sum_{k=1}^{N} \mathrm{ru}_k^n + \mathit{\Delta} x \mathit{\Delta} y(\tau_s - \tau_b)$$

- pre_step3d : Predictor for u,v,Hz,t

$$\mathrm{Hz}^{n+1/2} = (1/2+\gamma)\,\mathrm{Hz}^n + (1/2-\gamma)\,\mathrm{Hz}^{n-1} - (1-\gamma)\Delta t \cdot [\mathsf{div}_h\ \mathrm{HUon}^n + \partial_z\ \mathrm{W}^n]$$

Horizontal advection for tracers

$$t^{n+1/2} = (1/2+\gamma)\,\mathrm{Hz}^n t^n + (1/2-\gamma)\,\mathrm{Hz}^{n-1} t^{n-1} - (1-\gamma)\Delta t \cdot \mathsf{div}_h\ (\mathrm{HUon}^n t^n)$$

Vertical advection for tracers

$$t^{n+1/2} = \frac{1}{\mathrm{Hz}^{n+1/2}} \left[t^{n+1/2} - (1-\gamma)\Delta t \cdot \partial_z\,(W^n t^n)\right]$$

$\Rightarrow$ final value $t^{n+1/2}$

Advance u at time $n + 1/2$

$$u^{n+1/2} = \frac{1}{\mathrm{Hz}^{n+1/2}} \left[(1/2+\gamma)\,\mathrm{Hz}^n u^n + (1/2-\gamma)\,\mathrm{Hz}^{n-1} u^{n-1} - (1-\gamma)\Delta t \cdot (\mathrm{ru}^n)\right]$$

$\Rightarrow$ temporary value of $u^{n+1/2}$ before correction by the 2D

- → Apply boundary conditions for $t^{n+1/2}$ et $u^{n+1/2}$ (t3dbc,u3dbc,v3dbc)
- → Initialisation of the free-surface for the barotropic mode

$$\zeta_0 \longleftarrow \langle \zeta \rangle^n \, (= \mathrm{Zt\_avg1})$$

- uv3dmix : Compute lateral viscosity for $\mathrm{Hz}^n u^n$ at time n and add the vertical integral to rufrc^n

step2D_thread()

Barotropic time loop

→ AB3-extrapolation at time $m+1/2$

$$\begin{aligned}
D^{m+1/2} &= (3/2+\beta)\,\mathrm{zeta}^{m} - (1/2+2\beta)\,\mathrm{zeta}^{m-1} + \beta\mathrm{zeta}^{m-2} + H && (= \mathrm{Drhs})\\
\overline{u}^{m+1/2} &= (3/2+\beta)\,\mathrm{ubar}^{m} - (1/2+2\beta)\,\mathrm{ubar}^{m-1} + \beta\mathrm{ubar}^{m-2} && (= \mathrm{urhs})\\
(\Delta y)\overline{U}^{m+1/2} &= \Delta y\ D^{m+1/2}\ \overline{u}^{m+1/2} && (\mathrm{DUon} = \Delta y\ \text{Drhs}\ \mathrm{urhs})\\
\zeta^{m+1} &= \zeta^{m} + (\Delta\tau/\Delta y)\ \mathrm{div}_h\ \mathrm{DUon}^{m+1/2} && (= \mathrm{zeta\_new})\\
D^{m+1} &= \zeta^{m+1} + H && (= \mathrm{Dnew})
\end{aligned}$$

→ AM4-interpolation of the free-surface at time $m+1/2$

$$\zeta^{\star} = \alpha_0\zeta^{m+1} + \alpha_1\zeta^{m} + \alpha_2\zeta^{m-1} + \alpha_3\zeta^{m-2} \quad (= \mathrm{zwrk} = \mathrm{rzeta};\mathrm{rzeta2} = (\zeta^{\star})^2)$$

→ Boundary conditions on ζ^{m+1} (zetabc)

→ Time filtering ($\langle\zeta\rangle^{n+1} = \sum_m a_m \zeta^m, \left\langle\left\langle\overline{U}\right\rangle\right\rangle^{n+1/2} = \sum_m b_m \overline{U}^{m+\frac{1}{2}}$)

$$\begin{aligned}
\langle\zeta\rangle^{n+1} &= \langle\zeta\rangle^{n+1} + a_m\zeta^{m+1} && (= \mathrm{Zt\_avg1})\\
\left\langle\left\langle\overline{U}\right\rangle\right\rangle^{n+1/2} &= \left\langle\left\langle\overline{U}\right\rangle\right\rangle^{n+1/2} + \Delta y\ b_m\ \overline{U}^{m+1/2} && (= \mathrm{DU\_avg2})
\end{aligned}$$

→ Compute the right-hand-side $\mathrm{rubar}^{m+1/2}$:

- Pressure gradient ($\mathrm{rubar}^{m+1/2} = gH\partial_x\zeta^\star + (g/2)\partial_x\ \zeta^{\star\ 2}$)
- Horizontal advection
- Coriolis term
- Viscosity (optional)
- Bottom friction

→ At the first 2D time-step : extrapolation of the forcing rufrc at time $n + 1/2$

$$\begin{aligned}\mathrm{rufrc}^{n+1/2} &= (3/2+\delta)(\mathrm{rufrc}^n - \mathrm{rubar}^n) + \delta\ \mathrm{rufrc}^{n-1} - (1/2 + 2\delta)\ \mathrm{rufrc}^{n-2} \\ \mathrm{rufrc}^n &= \mathrm{rufrc}^n - \mathrm{rubar}^n \qquad (= \mathrm{rufrc\_bak(nstp)})\end{aligned}$$

→ Finalize the computation of the barotropic momentum

$$\begin{aligned}\overline{U}^{m+1} &= \overline{U}^m + \Delta\tau\ (\mathrm{rubar}^{m+1/2} + \mathrm{rufrc}^{n+1/2}) && (= \mathrm{DUnew}) \\ \overline{u}^{m+1} &= \mathrm{DUnew}/D^{m+1} && (= \mathrm{ubar(knew)}) \\ \langle\overline{U}\rangle^{n+1} &= \langle\overline{U}\rangle^{n+1} + \Delta y\ a_m\ \overline{U}^{m+1} && (= \mathrm{DU\_avg1})\end{aligned}$$

→ Boundary conditions on $\overline{u}^{m+1}$ (u2dbc, v2dbc)

End of the barotropic loop

step3D_uv_thread()

- set_depth : update of the grid (i.e. $\mathrm{z\_r}^{n+1}$, $\mathrm{z\_w}^{n+1}$, Hz^{n+1}) via $\langle\zeta\rangle^{n+1}$

- set_HUV2 : Correction of $u^{n+1/2}$ to ensure that

$$\sum_{k=1}^{N} \mathrm{Hz}^{n+1}\ \Delta y\ u^{n+1/2} = \mathrm{DU\_avg2}$$

$\Rightarrow$ final value of $u^{n+1/2}$

$$\mathrm{HUon}^{n+1/2} = \mathrm{Hz}^{n+1}\ \Delta y\ u^{n+1/2}$$

- omega : Computation of the volumetric flux $W^{n+1/2}$ (continuity equation)
- rho_eos : Update the density perturbation $\rho'^{\,n+1/2} = \rho_{\mathrm{EOS}}(t^{n+1/2})$
- prsgrd : Computation of the HPG $\mathrm{ru}_k^{n+1/2} = \left.\dfrac{\partial \mathrm{P}_k^{n+1/2}}{\partial x}\right|_z$

- rhs3d : Computation of the rhs $\mathrm{ru}^{n+1/2}$ for 3D momentum equations

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- step3d_uv1 : Corrector for u

$$u^{n+1} = u^n + \Delta t\,(\mathrm{ru}^{n+1/2})$$

- step3d_uv2 :
  - Resolution of a tridiagonal system for implicit vertical viscosity
  - Correction of u^{n+1} to ensure that

$$\sum_{k=1}^{N} \Delta y\ \mathrm{Hz}^{n+1}\ u^{n+1} = \mathrm{DU\_avg1}$$

  - Boundary conditions for u^{n+1} (u3dbc, v3dbc)
    $\Rightarrow$ final value of u^{n+1}

  - Initialisation of $\bar{u}$ for the barotropic integration at the next time-step

$$\bar{u}_0 \leftarrow \frac{\mathrm{DU\_avg1}}{\sum_{k=1}^{N} \Delta y\ \mathrm{Hz}_k^{n+1}}$$

- Computation of the volumetric flux centered at time $n + 1/2$

$$(\mathrm{Hz}\ u)^\star = \frac{3}{4}\mathrm{HUon}^{n+1/2} + \frac{\mathrm{Hz}^{n+1}}{8}(u^{n+1} + u^n)$$

- Correction of $(\mathrm{Hz}\ u)^\star$ to ensure that

$$\sum_{k=1}^{N} \Delta y\ (\mathrm{Hz}\ u)^\star = \mathrm{DU\_avg2}$$

- Update the horizontal volumetric fluxes at time $n + 1/2$

$$\mathrm{HUon}^{n+1/2} = \Delta y\ (\mathrm{Hz}\ u)^\star$$

step3D_t_thread()

- omega : Compute vertical volumetric flux $W^{n+1/2}$ via the continuity equation
- step3d_t : Corrector for t

$$t^{n+1} = \mathrm{Hz}^n t^n - \Delta t \ \mathrm{div}\left(\mathrm{HUon}^{n+1/2} t^{n+1/2}\right)$$

$$t^{n+1} = t^{n+1} - \Delta t \ \mathrm{div}\left(\mathrm{W}^{n+1/2} t^{n+1/2}\right)$$

→ Resolution of the tridiagonal system for the implicit vertical diffusion
→ Boundary conditions for t^{n+1} (t3dbc)

⇒ final value of t^{n+1}

5

Future evolutions of the CROCO time-stepping

Critical comments on current time-stepping

LFAM3

- significant loss of stability when using odd-ordered advection schemes
- inadequate for the implicit treatment of bottom drag
- AM3 interpolation has negative weights → no positive-definiteness
  - pb for layer thicknesses advection and wetting & drying
  - pb for biogeochemical tracers
- Existence of a computational mode which is not always efficiently damped
- Stability with lagrangian treatment of the vertical direction (remapping schemes)
- Complex interfacing with external modules

Generalized Forward-Backward

- Cold restart at each barotropic integration

Future evolutions

- Transition toward a Runge–Kutta framework
  - Absence of computational mode
  - RK + forward-backward feddbacks for internal waves
  - Easier implementation of time refinement with Agrif
  - Compatible with TVD or monotonicity-preserving advection schemes
  - Possibility to derive RK-based coupled space-time schemes for advection
- Design of an Arbitrary Lagrangian Eulerian vertical coordinate