Hydrostatic CROCO time-stepping flow chart

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pre_step3D_thread()

- rho_eos :

  → Compute density at time n

$$\rho'^{\,n} = \rho_{\mathrm{EOS}}(T^n, S^n)$$

  → Compute Brunt-Väisälä frequency

$$\mathrm{bvf}^n(\rho'^{\,n}, z)$$

- set_HUV : Compute horizontal volumetric fluxes (i.e. $U^n = \Delta y \Delta z^n u^n$)

$$\mathrm{HUon}_k^n = \mathrm{Hz}_k^n \; \Delta y \; \mathrm{u}_k^n$$

- omega : Compute vertical volumetric flux via the continuity equation

$$\mathrm{W}_{k+1/2}^n = -\sum_{k'=1}^{k} (\mathrm{div\ HUon})_{k'}^n + \frac{z_{k+1/2}^n - H}{\zeta^n - H} \sum_{k=1}^{N} (\mathrm{div\ HUon})_k^n$$

- prsgrd : Compute the pressure gradient and add it to the 3D right-hand-side ru

$$\mathrm{ru}_k = \left.\frac{\partial \mathrm{P}_k^n}{\partial x}\right|_z$$

- rhs3d : Compute the right-had-side at time n for momentum equations

  → Coriolis term

$$\mathrm{ru}^n = \mathrm{ru}^n + \mathrm{Coriolis}$$

→ Horizontal advection

$$\text{ru}^n = \text{ru}^n + \text{2D advection}$$

→ Vertical advection

$$\text{ru}^n = \text{ru}^n + \text{vertical advection}$$

Compute the forcing term for the 2D (barotropic) momentum equations (add the difference between the surface and bottom stress)

$$\text{rufrc}^n = \sum_{k=1}^{N} \text{ru}_k^n + \Delta x \Delta y (\tau_s - \tau_b)$$

- pre_step3d : Predictor step for u,v,Hz,t

$$\text{Hz}^{n+1/2} = (1/2 + \gamma)\, \text{Hz}^n + (1/2 - \gamma)\, \text{Hz}^{n-1} - (1-\gamma)\Delta t \cdot [\text{div}_h\ \text{HUon}^n + \partial_z\ \text{W}^n]$$

Horizontal advection for tracers

$$t^{n+1/2} = (1/2 + \gamma)\, \text{Hz}^n t^n + (1/2 - \gamma)\, \text{Hz}^{n-1} t^{n-1} - (1-\gamma)\Delta t \cdot \text{div}_h\ (\text{HUon}^n t^n)$$

Vertical advection for tracers

$$t^{n+1/2} = \frac{1}{\text{Hz}^{n+1/2}} \left[t^{n+1/2} - (1-\gamma)\Delta t \cdot \partial_z \left(W^n t^n \right) \right]$$

⇒ final value for $t^{n+1/2}$

Advance u to time $n + 1/2$

$$u^{n+1/2} = \frac{1}{\text{Hz}^{n+1/2}} \left[(1/2 + \gamma)\, \text{Hz}^n u^n + (1/2 - \gamma)\, \text{Hz}^{n-1} u^{n-1} - (1-\gamma)\Delta t \cdot (\text{ru}^n) \right]$$

⇒ provisional value for $u^{n+1/2}$ before barotropic correction

Warning : $u^{n-1} \longleftarrow u^n\ \text{Hz}^n$ (i.e. u(indx) = u(nstp)×Hz^n)

→ Boundary conditions for $t^{n+1/2}$ and $u^{n+1/2}$ (t3dbc,u3dbc,v3dbc)

→ Initialize the free-surface for the barotropic mode

$$\zeta_0 \longleftarrow \langle \zeta \rangle^n \ (= \text{Zt\_avg1})$$

- uv3dmix : Compute lateral viscosity for $\mathrm{Hz}^n u^n$ (i.e. u(indx)) at time n and integrate it vertically in rufrc^n.

step2D_thread()

- step2d : Barotropic time increment $= \Delta\tau$

Barotropic step loop

→ AB3 extrapolation at time $m+1/2$

$$
\begin{aligned}
D^{m+1/2} &= (3/2+\beta)\,\mathrm{zeta}^m - (1/2+2\beta)\,\mathrm{zeta}^{m-1} + \beta\mathrm{zeta}^{m-2} + H && (=\mathrm{Drhs})\\
\overline{u}^{m+1/2} &= (3/2+\beta)\,\mathrm{ubar}^m - (1/2+2\beta)\,\mathrm{ubar}^{m-1} + \beta\mathrm{ubar}^{m-2} && (=\mathrm{urhs})\\
(\Delta y)\overline{U}^{m+1/2} &= \Delta y\; D^{m+1/2}\;\overline{u}^{m+1/2} && (\mathrm{DUon} = \Delta y\ \mathrm{Drhs\ urhs})\\
\zeta^{m+1} &= \zeta^m + (\Delta\tau/\Delta y)\ \mathrm{div}_h\ \mathrm{DUon}^{m+1/2} && (=\mathrm{zeta\_new})\\
D^{m+1} &= \zeta^{m+1} + H && (=\mathrm{Dnew})
\end{aligned}
$$

→ AM4 interpolation of free-surface at time $m+1/2$

$$
\zeta^\star = \alpha_0\zeta^{m+1} + \alpha_1\zeta^m + \alpha_2\zeta^{m-1} + \alpha_3\zeta^{m-2} \quad (=\mathrm{zwrk}=\mathrm{rzeta};\mathrm{rzeta2}=(\zeta^\star)^2)
$$

→ Boundary conditions for ζ^{m+1} (zetabc)

→ Time filtering ($\langle\zeta\rangle^{n+1} = \sum_m a_m\zeta^m$, $\langle\langle\overline{U}\rangle\rangle^{n+1/2} = \sum_m b_m\overline{U}^{m+\frac{1}{2}}$)

$$
\begin{aligned}
\langle\zeta\rangle^{n+1} &= \langle\zeta\rangle^{n+1} + a_m\zeta^{m+1} && (=\mathrm{Zt\_avg1})\\
\langle\langle\overline{U}\rangle\rangle^{n+1/2} &= \langle\langle\overline{U}\rangle\rangle^{n+1/2} + \Delta y\; b_m\;\overline{U}^{m+1/2} && (=\mathrm{DU\_avg2})
\end{aligned}
$$

→ Compute the right-hand-side $\mathrm{rubar}^{m+1/2}$:

- Pressure gradient ($\mathrm{rubar}^{m+1/2} = gH\partial_x\zeta^\star + (g/2)\partial_x\,{\zeta^\star}^2$)
- Horizontal advection
- Coriolis term
- Viscosity (optionnel)
- Bottom friction

→ At the first 2D time-step : extrapolate the forcing[1] rufrc at time $n + 1/2$

$$\begin{aligned} \text{rufrc}^{n+1/2} &= (3/2+\delta)(\text{rufrc}^n - \text{rubar}^n) + \delta\ \text{rufrc}^{n-1} - (1/2+2\delta)\ \text{rufrc}^{n-2} \\ \text{rufrc}^n &= \text{rufrc}^n - \text{rubar}^n \qquad (= \text{rufrc\_bak(nstp)}) \end{aligned} \tag{1}$$

→ Finalize the barotropic computation

$$\begin{aligned} \overline{U}^{m+1} &= \overline{U}^m + \Delta\tau\ (\text{rubar}^{m+1/2} + \text{rufrc}^{n+1/2}) && (= \text{DUnew}) \\ \overline{u}^{m+1} &= \text{DUnew}/D^{m+1} && (= \text{ubar(knew)}) \\ \langle\overline{U}\rangle^{n+1} &= \langle\overline{U}\rangle^{n+1} + \Delta y\ a_m\ \overline{U}^{m+1} && (= \text{DU\_avg1}) \end{aligned}$$

→ Boundary conditions for $\overline{u}^{m+1}$ (u2dbc, v2dbc)

end barotropic step loop

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step3D_uv_thread()

- set_depth : : update the vertical grid (i.e. z_r^{n+1}, z_w^{n+1}, Hz^{n+1}) via $\langle\zeta\rangle^{n+1}$

- set_HUV2 : Correction of $u^{n+1/2}$ to ensure that

$$\sum_{k=1}^{N} \text{Hz}^{n+1}\ \Delta y\ u^{n+1/2} = \text{DU\_avg2}$$

⇒ final value for $u^{n+1/2}$

$$\text{HUon}^{n+1/2} = \text{Hz}^{n+1}\ \Delta y\ u^{n+1/2}$$

- omega : Update the vertical volumetric flux $W^{n+1/2}$ via the continuity equation

[1] This term corresponds to the *barocline-to-barotrope forcing term* which is the difference between the rhs obtained using barotropic variables and the one obtained by vertically integrating the 3D rhs.

- rho_eos : Update the density perturbation ${\rho'}^{n+1/2} = \rho_{\mathrm{EOS}}(t^{n+1/2})$
- prsgrd : Compute the horizontal pressure gradient

$$\mathrm{ru}_k^{n+1/2} = \left.\frac{\partial \mathrm{P}_k^{n+1/2}}{\partial x}\right|_z$$

- rhs3d : Compute the right-hand-side $\mathrm{ru}^{n+1/2}$ for 3D momentum

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- step3d_uv1 : Corrector step for u

$$u^{n+1} = u^n + \Delta t\ (\mathrm{ru}^{n+1/2})$$

- step3d_uv2 :
  - Solve tridiagonal system for implicit vertical viscosity
  - Correction of u^{n+1} to ensure that

$$\sum_{k=1}^{N} \Delta y\ \mathrm{Hz}^{n+1}\ u^{n+1} = \mathrm{DU\_avg1}$$

  - Boundary conditions for u^{n+1} (u3dbc, v3dbc)
    $\Rightarrow$ final value of u^{n+1}
  - Initialization of $\bar{u}$ for the next barotropic integration

$$\bar{u}_0 \leftarrow \frac{\mathrm{DU\_avg1}}{\sum_{k=1}^{N} \Delta y\ \mathrm{Hz}_k^{n+1}}$$

  - Compute volumetric fluxes centered at time $n + 1/2$

$$(\mathrm{Hz}\ u)^\star = \frac{3}{4}\mathrm{HUon}^{n+1/2} + \frac{\mathrm{Hz}^{n+1}}{8}(u^{n+1} + u^n)$$

  - Correction of $(\mathrm{Hz}\ u)^\star$ to ensure that

$$\sum_{k=1}^{N} \Delta y\ (\mathrm{Hz}\ u)^\star = \mathrm{DU\_avg2}$$

– Update the horizontal volumetric fluxes at time $n + 1/2$

$$\mathrm{HUon}^{n+1/2} = \Delta y \ (\mathrm{Hz}\ u)^{\star}$$

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step3D_t_thread()

- omega : Compute the vertical volumetric flux $W^{n+1/2}$ via the continuity equation
- step3d_t : Corrector step for t

$$t^{n+1} = \mathrm{Hz}^{n} t^{n} - \Delta t \ \mathrm{div} \left(\mathrm{HUon}^{n+1/2} t^{n+1/2} \right)$$

$$t^{n+1} = t^{n+1} - \Delta t \ \mathrm{div} \left(\mathrm{W}^{n+1/2} t^{n+1/2} \right)$$

→ Solve tridiagonal system for implicit vertical diffusion

→ Boundary conditions for t^{n+1} (t3dbc)

⇒ final value of t^{n+1}